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Frequency response of bodies with combined convective and radiative heat transfer

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Abstract

An analytical model for solving the problem of frequency response of bodies with combined convective and radiative heat transfer is presented. The issue of two conflicting types of conclusions existing in the literature regarding the effect of radiative heat transfer on the frequency response of bodies has been addressed and explained. It is shown how both types of conclusions are possible depending upon the type of assumptions involved in the solution of the problem or the scope and limitations of experimental observations. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The problem of radiative heat transfer combined with conductive and convective heat transfer has been the subject of study for years. After Heisler [1] solved the simple transient convective heat transfer problem, the first paper on the additional effects of radiative heat transfer was reported by Scadron and Warshawsky [2] in 1952. Since then there have been numerous papers [3–11] on the topic using analytical, numerical and/or experimental techniques to ascertain the effects of radiative heat transfer on the temperature frequency response of various devices. Scadron and Warshawsky [2] followed a typical engineering approach for their specific problem of finding the time constant for very hot fine wire thermocouples. They decided to linearize the boundary condition by making the assumption that the temperature of the fine wire thermocouple was close to the gas temperature and thus the thermocouple time constant was modified by a factor of $\{1+(4\varepsilon\sigma T_g^3)/h\}$. This is like deciding to use a linearized radiative heat transfer coefficient and combine it with the convective heat transfer coefficient, as suggested by Eckert and Drake [3] and by Schödel and Grigull [4]. While their logic and conclusions in the paper were correct given the assumption that the wire temperature was very near the gas temperature, this is generally not true.

Ayers [5] used a finite difference technique to make general charts. He divided a fine wire thermocouple in 21 discrete elements and showed the results for the temperature history in the wire as a function of five

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Nomenclature

A	surface area
c_v	specific heat at constant volume
err	multiplicative error in evaluation of Bessel functions
h	convective heat transfer coefficient
I	imaginary part of the response equation
i	$\sqrt{-1}$
k	thermal conductivity
K	an analytically derived constant
m	mass
r	radius
R	real part of the response equation
t	time
T	temperature.

Greek symbols

α	thermal diffusivity/phase lag
ε	emissivity
ρ	normalized angular frequency
σ	Stefan–Boltzman constant
ω_0	fundamental angular frequency (rad/s).

Subscripts

b	convectively and radiatively heated body
c	combined convective and radiative heat transfer coefficient
e	equilibrium
g	gas
j	frequency index
k	frequency index/index of term in Taylor expansion
m	frequency index
n	frequency index
q	frequency index
r	radiative heat transfer coefficient
s	evaluated at the surface or of the surface
w	wall
0	average.

non-dimensional variables, non-dimensionalized radiation intensity being one of them. His charts showed that the low frequency response was unity and that the time constant was affected by radiation through a T^3 term. Sucec and Kumar [6] used a finite difference solution to make general charts which showed a low frequency response of unity and that the time constant was affected by radiation. He compared his 'exact' numerical results to analytical results using a linearized radiation term in the boundary condition and the results did not agree. He concluded that using the Heisler charts with a combined 'convective–radiative' heat transfer coefficient can produce serious errors. Janata [7] derived analytical results by linearizing the boundary condition by using an equilibrium temperature, T_e, at which there is no 'radiative–convective' heat flux. He concluded that the low frequency response was unity and that the time constant varied as $(1+KT_e^3)$. Elmore et al. [8] used a finite difference technique to determine the frequency response of a thermocouple. They varied the strength of the radiative term by setting the emissivity to either zero or one over the range of frequencies, 20, 100 and 1000 Hz. It may be noted that all these frequencies are above the frequency (about 1 Hz) where attenuation of the response due to thermal lag occurs. No results were shown for the cases of asymptotically low frequencies and the conclusion was that radiative heat transfer had a minimal effect on the time constant. Shaibi et al. [9] reported from their experimental work that the time constant varies in an inverse reciprocal relationship with T_b^3. They measured the temperature of a fine wire thermocouple as the current through it was modulated. As the frequency of the current oscillations was increased, the magnitude of the temperature oscillations of the thermocouple was normalized with respect to the low frequency magnitude of the temperature oscillations. It was shown that the effect of radiation is to decrease the thermocouple time constant. Malcorps [10,11] published two papers on his work in the area of determining the frequency response of radiative heat flux sensors used to measure the solar radiation flux. He showed that the ability of the sensor to follow changes in gas temperature should decrease with increasing probe temperature and the response at high frequencies should vary as it does in the linear case; that is, the time constant is unaffected by the presence of radiative heat transfer. In other words, the frequency response for asymptotically low frequencies is affected by radiation but the time constant is not. This is a significantly different result from other authors.

With all of this work it is interesting to note that, with the only exception of Malcorps [10,11] work, the two types of conclusions in the literature are at odds with each other. The first type of conclusion is that the effect of radiative heat transfer has no effect on the frequency response. The second type of conclusion is that the radiative time constant is affected by radiation through a T^3 term. It would seem from the literature that on the issue of the effect of radiation the answer could either be that it is apparent in the time constant

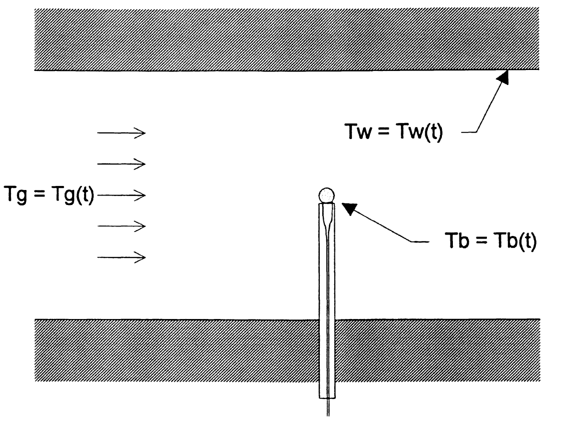

Fig. 1. Typical temperature sensor to be modeled as a lumped mass.

or it could be that it has no effect. The analytical and numerical work in this paper is aimed at explaining the fact that both types of conclusions are possible depending on the type of assumptions taken in the analysis or the scope of the experiment performed. It successfully explains the discrepancy existing in the literature and makes neither type of conclusion. The results presented here are in agreement with the work on Malcorps [10,11] who has reported similar conclusions.

2. Existing analytical models

Most of the analytical methods presented in the literature have followed the approach of determining the sensor's response to changes in gas temperature, given a variation in gaseous temperature. The physical arrangement of typical cases examined in the literature is shown in Fig. 1. Assuming a view factor of unity and neglecting gas phase radiative heat transfer, modeling of the sensing element as a lumped mass leads to solving the differential equation

$$h(T_g(t) - T_b(t)) = \frac{mc_v}{A}\frac{dT_b(t)}{dt} + \varepsilon\sigma(T_w^4 - T_b^4(t)) \tag{1}$$

This equation represents an implicit dependence of the body's temperature on gas temperature. If one assumes the gas temperature as the known boundary condition, an a priori linearization of the radiation terms is required such that the problem can be solved using analytical techniques. In the literature there are three primary methods for linearizing Eq. (1). The first method, as Scadron and Warshawsky [2] proposed, is to assume that the sensor temperature follows the gas temperature, $[T_g(t) - T_g(t)]/T_g(t) \approx 1$. The second method is to assume that the gas temperature and the body's temperature are close to the wall temperature, i.e. $T_b(t) \approx T_w(t) \approx T_b(t)$, as Sbaibi et al. [9] proposed. The third method proposed by Janata [7] is to assume that the radiative effects are best represented by an equilibrium temperature, T_e, at which radiative and convective heat transfer balance. By using any of these assumptions Eq. (1) can be linearized as

$$h_c(T_g(t) - T_b(t)) = \frac{mc_v}{A}\frac{dT_b(t)}{dt} \tag{2}$$

resulting in a modified heat transfer coefficient (h_c) where

$$h_c = (h + 4\varepsilon\sigma T_b^3) \quad \text{or}\ (h + 4\varepsilon\sigma T_g^3) \quad \text{or}\ (h + 4\varepsilon\sigma T_e^3)$$

depending on the assumption used. The transfer function is then found to be

$$\frac{T_b}{T_g} = \frac{1}{\sqrt{1 + \left(\frac{mc_v}{h_c A}\omega\right)^2}} \tag{3}$$

with a phase lag of

$$\alpha_g - \alpha_b = \tan^{-1}\left(\frac{mc_v}{h_c A}\omega\right) \tag{4}$$

This modification of the heat transfer coefficient then leads to the conclusion that the effect of radiation is to increase the ability of the sensor to follow changes in gas temperature. That is, an increase in the convective heat transfer coefficient or an increase in the radiative term results in an improved temperature response of the body in the flow.

3. Proposed solution model

The general analytical method validated in this paper begins with stating the boundary condition of the body to be given a priori. This is the central distinction of the work presented in this paper from the other work reported in the literature. By using the surface temperature of the body as the known boundary condition the gas temperature becomes an explicit function of the body temperature. The assumption that the temperature of the body is the given variable leads to a result that explains all the differences in the literature and agrees with the earlier reported analytical as well as numerical results. Thus in this paper, it is not the assumptions of previous work that are called into question but it is intended to explain when it is better to linearize the radiation terms leading to the maximum applicability of the solution. As it will be shown by delaying the linearization of the radiation terms towards the end of the derivation, the form of the equations is slightly different leading to a markedly different result.

For the variation in the body's surface temperature any physically realistic function can be chosen as long as it is a periodic and bounded piece-wise differential function [12]. This makes a Fourier series representation of the boundary condition possible. Then using Fourier's heat conduction equation, the equation describing the temperature fluctuations inside the body is derived. Once both the characteristics of the temperature at the surface and the variations in the wall temperature are known, the result is substituted into the non-linear boundary condition yielding an explicit non-linear equation. If one subtracts out the steady state solution then a non-linear equation for the unsteady heat transfer is left. By assuming that the fluctuations in temperatures of the body and the wall are smaller than their respective average temperatures, non-linear higher-order terms can be neglected. The problem is then easily solved for the ratio of the gas temperature fluctuations to the body surface temperature fluctuations (i.e. for the frequency response). It may be noted that this method is general in the sense that, as long as the form of the temperature fluctuations inside the body can be solved and a boundary condition can be written, the only assumption to be made is that the body's temperature fluctuations are small relative to its average temperature. As is shown in subsequent sections, this is true regardless of whether the non-linear radiation terms are dominant or not.

4. Example cases

4.1. Constant wall temperature ($T_w(t) = T_{w0}$) and lumped mass model

First a function describing the surface temperature of the body is taken. For the purpose of general applicability a Fourier function is chosen. Any bounded piecewise differential temperature fluctuation of the element over a finite time can be described by [12].

$$T_b(t) = T_{b0} + \sum_{q=1}^{\infty} T_{bq}\, e^{i(q\omega_0 t + \alpha_{bq})} \tag{5}$$

This is a Fourier series in magnitude and phase form. T_{bq} is a real number and the purpose of α_{bq} is to force the coefficients of the complex terms to be real and to account for the phase difference between the temperatures of the body and the gas. Phase of the gas temperature fluctuations are referenced to 0. Assuming that the Biot number is small (<0.1), the object can be modeled as a lumped mass [13]. Therefore, assuming no gas phase radiation and a view factor of 1 (see Fig. 1), the governing differential equation is

$$T_g(t) = T_b(t) + \frac{mc_v}{hA}\frac{dT_b}{dt} - \frac{\varepsilon\sigma}{h}(T_w^4 - T_b^4(t)) \tag{6}$$

Individual terms of the Fourier series being orthogonal, the linear terms are separable. The radiative term, however, is not linear and solutions for each individual frequency cannot be found separately. Therefore temperature fluctuations at one frequency can affect temperature fluctuations at other frequencies. To examine the coupling of the frequencies from Eq. (5) the fourth power of $T_b(t)$ is expressed as

$$\begin{aligned}
T_b^4(t) = {} & T_{b0}^4 + 4T_{b0}^3 \sum_{j=1}^{\infty} T_{bj}\, e^{i(j\omega_0 t + \alpha_{bj})} \\
& + 6T_{b0}^2 \left(\sum_{j=1}^{\infty} T_{bj}\, e^{i(j\omega_0 t + \alpha_{bj})}\right)\left(\sum_{k=1}^{\infty} T_{bk}\, e^{i(k\omega_0 t + \alpha_{bk})}\right) \\
& + 4T_{b0}\left(\sum_{j=1}^{\infty} T_{bj}\, e^{i(j\omega_0 t + \alpha_{bj})}\right)\left(\sum_{k=1}^{\infty} T_{bk}\, e^{i(k\omega_0 t + \alpha_{bk})}\right) \\
& \times \left(\sum_{m=1}^{\infty} T_{bm}\, e^{i(m\omega_0 t + \alpha_{bm})}\right) + \left(\sum_{j=1}^{\infty} T_{bj}\, e^{i(j\omega_0 t + \alpha_{bj})}\right) \\
& \times \left(\sum_{k=1}^{\infty} T_{bk}\, e^{i(k\omega_0 t + \alpha_{bk})}\right)\left(\sum_{m=1}^{\infty} T_{m}\, e^{i(m\omega_0 t + \alpha_{bm})}\right) \\
& \times \left(\sum_{n=1}^{\infty} T_{bn}\, e^{i(n\omega_0 t + \alpha_{bn})}\right)
\end{aligned} \tag{7}$$

Multiplying the summations through, Eq. (7) becomes

$$\begin{aligned}
T_b^4(t) = {} & T_{b0}^4 + 4T_{b0}^3 \sum_{j=1}^{\infty} T_{bj}\, e^{i(j\omega_0 t + \alpha_{bj})} \\
& + 6T_{b0}^2 \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} T_{bj} T_{bk}\, e^{i((j+k)\omega_0 t + \alpha_{bj} + \alpha_{bk})} \\
& + 4T_{b0} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} T_{bj} T_{bk} T_{bm} \\
& \times e^{i((j+k+m)\omega_0 t + \alpha_{bj} + \alpha_{bk} + \alpha_{bm})} \\
& + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{bj} T_{bk} T_{bm} T_{bn} \\
& \times e^{i((j+k+m+n)\omega_0 t + \alpha_{bj} + \alpha_{bk} + \alpha_{bm} + \alpha_{bn})}
\end{aligned} \tag{8}$$

Substitution of Eq. (8) into the boundary condition [Eq. (6)] yields

$$T_g(t) = T_{b0} + \sum_{q=1}^{\infty} T_{bq}\, e^{i(q\omega_0 t + \alpha_{bq})} + \frac{mc_v}{hA}\sum_{q=1}^{\infty} T_{bq} iq\omega_0\, e^{i(q\omega_0 t+\alpha_{bq})} + \frac{\varepsilon\sigma}{h}\Bigg[T_{b0}^4 + 4T_{b0}^3 \sum_{j=1}^{\infty} T_{bj}\, e^{i(j\omega_0 t+\alpha_{bj})} + 6T_{b0}^2 \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} T_{bj}T_{bk}\, e^{i((j+k)\omega_0 t+\alpha_{bj}+\alpha_{bk})} + 4T_{b0}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} T_{bj}T_{bk}T_{bm} \times e^{i((j+k+m)\omega_0 t+\alpha_{bj}+\alpha_{bk}+\alpha_{bk}+\alpha_{bm})} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{bj}T_{bk}T_{bm}T_{bn} \times e^{i((j+k+m+n)\omega_0 t+\alpha_{bj}+\alpha_{bk}+\alpha_{bm}+\alpha_{bn})} - T_{w0}^4\Bigg] \tag{9}$$

If the amplitude of the fluctuating terms is set to 0, the constant terms can be seen to be

$$T_{g0} = T_{b0} + \frac{\varepsilon\sigma}{h}(T_{b0}^4 - T_{w0}^4) \tag{10}$$

Subtracting out the constant terms, the result is

$$\sum_{q=1}^{\infty} T_{gq}\, e^{i(q\omega_0 t+\alpha_{gq})} = \sum_{q=1}^{\infty} T_{bq}\, e^{i(q\omega_0 t+\alpha_{bq})} + \frac{mc_v}{hA}\sum_{q=1}^{\infty} iq\omega_0 T_{bq}\, e^{i(q\omega_0 t+\alpha_{bq})} + \frac{\varepsilon\sigma}{h}\Bigg[4T_{b0}^3\sum_{j=1}^{\infty} T_{bj}\, e^{i(j\omega_0 t+\alpha_{bj})} + 6T_{b0}^2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} T_{bj}T_{bk} \times e^{i((j+k)\omega_0 t+\alpha_{bj}+\alpha_{bk})} + 4T_{b0}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} T_{bj}T_{bk}T_{bm} \times e^{i((j+k+m)\omega_0 t+\alpha_{bj}+\alpha_{bk}+\alpha_{bm})} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{bj}T_{bk}T_{bm}T_{bn} \times e^{i((j+k+m+n)\omega_0 t+\alpha_{rmbj}+\alpha_{bk}+\alpha_{bm}+\alpha_{bn})}\Bigg] \tag{11}$$

To investigate the effect of the radiative term on the frequency response, a general single frequency is examined. By taking this single frequency to be the qth one, and accordingly setting the frequency index in all the terms to q, Eq. (11) becomes

$$T_{gq}\, e^{i(q\omega_0 t+\alpha_{gq})} = T_{bq}\, e^{i(q\omega_0 t+\alpha_{bq})} + \frac{mc_v}{hA} T_{bq} iq\omega_0\, e^{i(q\omega_0 t+\alpha_{bq})} + \frac{\varepsilon\sigma}{h}\Bigg[4T_{b0}^3 T_{bq}\, e^{i(q\omega_0 t+\alpha_{bq})} + 6T_{b0}^2\sum_{k=1}^{k<q} T_{bk}T_{b(q-k)}\, e^{i(q\omega_0 t+\alpha_{bq}+\alpha_{b(k-q)})} + 4T_{b0}\sum_{k=1}^{k<q-1}\sum_{m=1}^{m<q-k-1} T_{bk}T_{bm}T_{b(q-k-m)} \times e^{i(q\omega_0 t+\alpha_{bk}+\alpha_{bm}+\alpha_{b(q-k-m)})} + \sum_{k=1}^{k<q-2}\sum_{m=1}^{m<q-k-2}\sum_{n=1}^{n<q-k-m-2} \times T_{bk}T_{bm}T_{bn}T_{b(q-k-m-n)} \times e^{i(q\omega_0 t+\alpha_{bk}+\alpha_{bm}+\alpha_{b(q-k-m-n)})}\Bigg] \tag{12}$$

This shows that the gas phase temperature fluctuations are not only determined by the fluctuations of the temperature sensor at the frequency $q\omega_0$, but are also affected by the fluctuations at frequency components less than $q\omega_0$. By examining the last three terms in Eq. (12), it can be seen how the individual lower frequency temperature fluctuations affect the fluctuation at $q\omega_0$ by interacting with other lower frequency temperature fluctuations. This effect of frequency harmonics is commonly seen in nonlinear systems but usually it is not possible to write out their effects as in Eq. (12). It is possible in this case because the sensor is a passive element in the system. If the temperature of the sensor were to feed back into the gas temperature, then it would not be possible to explicitly write out the effect of each frequency component.

If Eq. (12) is divided out by the varying term, $e^{i(q\omega_0 t)}$, the amplitude of each term is given by

$$T_{gq}\, e^{i(\alpha_{gq})} = T_{bq}\, e^{i(\alpha_{bq})} + i\frac{mc_v}{hA} T_{bq} q\omega_0\, e^{i(\alpha_{bq})} + \frac{\varepsilon\sigma}{h}\Bigg[4T_{b0}^3 T_{bq} \times e^{i(\alpha_{bq})} + 6T_{b0}^2\sum_{k=1}^{k<q} T_{bk}T_{b(q-k)}\, e^{i(\alpha_{bq}+\alpha_{b(k-q)})} + 4T_{b0}\sum_{k=1}^{k<q-1}\sum_{m=1}^{m<q-k-1} T_{bk}T_{bm}T_{b(q-k-m)} \times e^{i(\alpha_{bk}+\alpha_{bm}+\alpha_{b(q-k-m)})} + \sum_{k=1}^{k<q-2}\sum_{m=1}^{m<q-k-2}\sum_{n=1}^{n<q-k-m-2} \times T_{bk}T_{bm}T_{bn}T_{b(q-k-m-n)}\, e^{i(\alpha_{bk}+\alpha_{bm}+\alpha_{b(q-k-m-n)})}\Bigg] \tag{13}$$

To obtain the transfer function, defined as the ratio of the fluctuations of the lumped mass temperature to the fluctuations in gas temperature, an approximation is to be made. Assuming that the coherent sum of the fluctuating terms are much smaller than the mean temperature, it is apparent that

$$4T_{b0}^3 T_{bq}\,e^{i(\alpha_{bq})} \gg 6T_{b0}^2 \sum_{k=1}^{k<q} T_{bk} T_{b(q-k)}\,e^{i(\alpha_{bq}+\alpha_{b(k-q)})} + 4T_{b0} \sum_{k=1}^{k<q-1} \sum_{m=1}^{m<q-k-1} T_{bk} T_{bm} T_{b(q-k-m)} \times e^{i(\alpha_{bk}+\alpha_{bm}+\alpha_{b(q-k-m)})} + \sum_{k=1}^{k<q-2} \sum_{m=1}^{m<q-k-2} \sum_{n=1}^{n<q-k-m-2} \times T_{bk} T_{bm} T_{bn} T_{b(q-k-m-n)}\,e^{i(\alpha_{bk}+\alpha_{bm}+\alpha_{bn}+\alpha_{b(q-k-m-n)})} \tag{14}$$

Thus Eq. (13) can be simplified to

$$T_{gq}\,e^{i(\alpha_{gq})} = T_{bq}\,e^{i(\alpha_{bq})} + i\frac{mc_v}{hA} T_{bq} q\omega_0\, e^{i(\alpha_{bq})} + \left(\frac{\varepsilon\sigma}{h} 4T_{b0}^3 T_{bq}\,e^{i(\alpha_{bq})}\right) \tag{15}$$

Eq. (15) is valid for all frequency components less than equal to $q\omega_0$. Thus, even if the approximation shown in Eq. (14) is not valid for $(q + 1)\omega_0$, that frequency component does not affect lower-order frequency components and the approximation is still valid for the frequency components less than $(q + 1)\omega_0$.

From Eq. (15) the frequency response, i.e. ratio of fluctuations in the gas temperature to those of the lumped mass, is

$$\frac{T_{bq}}{T_{gq}} = \frac{1}{\sqrt{\left(1+\left(\frac{4\varepsilon\sigma}{h} T_{b0}^3\right)\right)^2 + \left(\frac{mc_v}{hA} q\omega_0\right)^2}} \tag{16}$$

with the gas phase leading the lumped mass by

$$\alpha_{gq} - \alpha_{bq} = \tan^{-1}\left(\frac{\frac{mc_v}{hA} q\omega_0}{1+\frac{4\varepsilon\sigma}{h} T_{b0}^3}\right) \tag{17}$$

Thus we have the transfer function for a lumped mass in a stream with combined radiative and convective heat transfer.

If it is required to normalize the response such that it becomes unity at asymptotically low frequencies, it is simple to divide the frequency response by the frequency response in the limit of low frequencies such that the frequency response is

$$\frac{T_{bq}}{T_{gq}} = \frac{1}{\sqrt{1+\left(\frac{\frac{mc_v}{hA}}{\frac{4\varepsilon\sigma}{h} T_{b0}^3} q\omega_0\right)^2}} \tag{18}$$

It may be noted that Eq. (18) is identical to the theoretical result of Sbaibi et al. [9].

4.2. Varying wall temperature and lumped mass model

In general the wall temperature can be a function of time. Thus, the equation governing the heat transfer to a lumped mass becomes

$$T_g(t) = T_b(t) + \frac{mc_v}{h} \frac{dT_b}{dt} - \frac{\varepsilon\sigma}{h}\left(T_w^4(t) - T_b^4(t)\right) \tag{19}$$

Over a finite time any piecewise differentiable temperature fluctuation of the lumped mass can be described by a Fourier series as in Eq. (5).

$$T_w(t) = T_{w0} + \sum_{q=1}^{q=\infty} T_{wq}\,e^{i(q\omega_0 t+\alpha_{wq})} \tag{20}$$

Using a similar approach as stated in Eq. (14), if the coherent sum of the wall temperature oscillations are small compared to the mean wall temperature, it can be shown that

$$T_{w0} + \left(\sum_{q=1}^{q=\infty} T_{wq}\,e^{i(q\omega_0 t)}\right)^4 \approx T_{w0}^4 + 4T_{w0}^3 \sum_{q=1}^{q=\infty} T_{wq}\, e^{i(q\omega_0 t+\alpha_{wq})} \tag{21}$$

Substitution of this into Eq. (19) results in

$$T_{gq}\,e^{i(\alpha_{gq})} = T_{bq}\,e^{i(\alpha_{bq})} + i\frac{mc_v}{hA} T_{bq} q\omega_0\, e^{i(\alpha_{bq})} + \frac{\varepsilon\sigma}{h}\left(4T_{b0}^3 T_{bq}\,e^{i(\alpha_{bq})} - 4T_{w0}^3 T_{wq}\,e^{i(\alpha_{wq})}\right) \tag{22}$$

which yields the transfer function

$$\frac{T_{bq}}{T_{gq}} = \frac{1}{\sqrt{R^2+I^2}} \tag{23}$$

with the gas phase leading the lumped mass by

$$\alpha_{gq} - \alpha_{bq} = \tan^{-1}\left(\frac{\frac{mc_v}{hA} q\omega_0 + \frac{4\varepsilon\sigma}{h} T_{w0}^3 \frac{T_{wq}}{T_{bq}}\ \sin(\alpha_{wq}-\alpha_{bq})}{1+\frac{4\varepsilon\sigma}{h}\left(T_{b0}^3 \frac{T_{wq}}{T_{bq}}\ \cos(\alpha_{wq}-\alpha_{bq})\right)}\right) \tag{24}$$

where

$$R = 1 + \frac{4\varepsilon\sigma}{h}\left(T_{b0}^{3} - T_{w0}^{3}\frac{T_{wq}}{T_{bq}}\ \cos(\alpha_{wq} - \alpha_{bq})\right)$$

and

$$I = \frac{mc_v}{hA}q\omega_0 + \frac{4\varepsilon\sigma}{h}T_{w0}^{3}\frac{T_{wq}}{T_{bq}}\ \sin(\alpha_{wq} - \alpha_{bq})$$

Thus, we have the solution for the transfer function of a lumped mass with convective and radiative heat transfer assuming that the temperature oscillations of the lumped mass and that of the walls are much smaller than their respective mean temperatures. This assumption will be examined while discussing some of the results later.

If the radiation from the object and the wall are in phase ($\alpha_{wq} = \alpha_{bq}$) and the magnitude of the transient terms are equal ($T_{b0}^3 T_{bq} = T_{w0}^3 T_{wq}$), then Eq. (23) and (24) would reduce, respectively, to

$$\frac{T_{bq}}{T_{gq}} = \frac{1}{\sqrt{1 + \left(\frac{mc_v}{hA}q\omega_0\right)^2}} \tag{25}$$

and

$$\alpha_{gq} - \alpha_{bq} = \tan^{-1}\left(\frac{mc_v}{hA}q\omega_0\right) \tag{26}$$

4.3. Varying wall temperature and cylindrical mass model

Given that any cylindrical surface temperature, that can be expressed as a bounded piecewise differential temperature fluctuation over a finite time, can be expressed by

$$T_b(r_s, t) = T_{b0} + \sum_{q=1}^{q=\infty} T_{bq}\, e^{i(q\omega_0 t + \alpha_{bq})} \tag{27}$$

The governing differential equation for this case will be

$$\alpha\left(\frac{\partial^2 T_b}{\partial r^2} + \frac{1}{r}\frac{\partial T_b}{\partial r}\right) = \frac{\partial T_b}{\partial t} \tag{28}$$

subject to the boundary condition

$$h(T_g(t) - T_b(r_s, t)) + \varepsilon\sigma(T_w^4(t) - T_b^4(r_s, t)) = k\nabla T_b(r_s, t) \tag{29}$$

Assuming that the temperature fluctuations over an infinite time can be written as the product of two functions—one of position only and the other of time alone, one arrives at

$$\nabla T_b(\rho_s, t) = \sum_{q=1}^{q=\infty}\sqrt{\frac{q\omega_0}{2\alpha}}(\phi_1(\rho_{qs}) + i\phi_2(\rho_{qs}))T_{bq}\, e^{i(q\omega_0 t + \alpha_{bq})}$$

where

$$\rho_{qs} = \sqrt{\frac{q\omega_0}{\alpha}}r_s$$

$$\phi_1(\rho_{qs}) = \frac{ber_0(\rho_{qs})ber_1(\rho_{qs}) + ber_0(\rho_{qs})bei_1(\rho_{qs}) - bei_0(\rho_{qs})ber_1(\rho_{qs}) + bei_0(\rho_{qs})bei_1(\rho_{qs})}{ber_0^2(\rho_{qs}) + bei_0^2(\rho_{qs})}$$

and

$$\phi_2(\rho_{qs}) = \frac{-ber_0(\rho_{qs})ber_1(\rho_{qs}) + ber_0(\rho_{qs})bei_1(\rho_{qs}) - bei_0(\rho_{qs})ber_1(\rho_{qs}) + bei_0(\rho_{qs})bei_1(\rho_{qs})}{ber_0^2(\rho_{qs}) + bei_0^2(\rho_{qs})} \tag{30}$$

Substituting this result into Eq. (29) with approximations for the radiative terms similar to Eq. (14), and subtracting out the mean temperature terms, one arrives at

$$T_{gq}\, e^{i(\alpha_{gq})} = T_{bq}\, e^{i(\alpha_{bq})} + \frac{k}{h}\sqrt{\frac{q\omega_0}{2\alpha}}(\phi_1(\rho_{qs}) + i\phi_2(\rho_{qs}))T_{bq}\, e^{i(\alpha_{bq})} + \frac{\varepsilon\sigma}{h}(4T_{b0}^{3}T_{bq}\, e^{i(\alpha_{bq})} - 4T_{w0}^{3}T_{wq}\, e^{i(\alpha_{wq})}) \tag{31}$$

which results in the transfer function

$$\frac{T_{bq}}{T_{gq}} = \frac{1}{\sqrt{R^2 + I^2}} \tag{32}$$

where

$$R = 1 + \frac{4\varepsilon\sigma}{h}\left(T_{b0}^3 - T_{w0}^3 \frac{T_{wq}}{T_{bq}}\right.$$

$$\left.\cos(\alpha_{wq} - \alpha_{bq})\right) + \frac{k}{h}\sqrt{\frac{q\omega_0}{2\alpha}}\phi_1(\rho_{qs})$$

and

$$I = \frac{4\varepsilon\sigma}{h} T_{w0}^3 \frac{T_{wq}}{T_{bq}} \sin(\alpha_{wq} - \alpha_{bq}) + \frac{k}{h}\sqrt{\frac{q\omega_0}{2\alpha}}\phi_2(\rho_{qs})$$

with the gas phase leading the lumped mass by

$$(\alpha_{gq} - \alpha_{bq}) = \tan^{-1}\left(\frac{\frac{4\varepsilon\sigma}{h} T_{w0}^3 \frac{T_{wq}}{T_{bq}} \sin(\alpha_{wq} - \alpha_{bq}) + \frac{k}{h}\sqrt{\frac{q\omega_0}{2\alpha}}\phi_2(\rho_{qs})}{1 + \frac{4\varepsilon\sigma}{h}\left(T_{b0}^3 - T_{w0}^3 \frac{T_{wq}}{T_{bq}} \cos(\alpha_{wq} - \alpha_{bq})\right) + \frac{k}{h}\sqrt{\frac{q\omega_0}{2\alpha}}\phi_1(\rho_{qs})}\right) \tag{33}$$

It may be noted that for a large non-dimensional radius, $\rho_{qs} \to \infty$, Eqs. (32) and (33) reduce identically to those presented by Malcorps for planar 1-D heat flux meters.

5. Numerical evaluation of the response of a lumped mass

The numerical results for the lumped mass were found by solving Eq. (1) for the gas temperature as a function of the lumped mass temperature and the wall temperature. The resulting time varying gas temperature was found by substituting an equation with an average term and a single sinusoidal term for the time varying lumped mass temperature, and similarly substituting an equation with an average term and a sinusoidal term for the wall temperature. That is,

$$T_b(t) = T_{b0} + T_{bq} \sin(\omega_q t) \quad \text{and}$$
$$T_w(t) = T_{w0} + T_{wq} \sin(\omega_q t) \tag{34}$$

Thus, Eq. (1) gives

$$T_g(t) = T_{b0} + T_{bq} \sin(\omega_q t) + \frac{mc_v}{hA}\omega_q T_{bq}$$

$$\cos(\omega_q t) - \frac{\varepsilon\sigma}{h}((T_{b0} + T_{bq}$$

$$\sin(\omega_q t))^4 - (T_{w0} + T_{wq} \sin(\omega_q t))^4) \tag{35}$$

6. Results

The three analytical results derived in the previous section, two for the lumped mass and one for the cylindrical body, show the general method of solution for obtaining the frequency response of convectively and radiatively heated bodies. In this section some sample results are presented to show the general validity of the analytical method presented above in Sections 3 and 4. It is also shown how the discrepancies existing in the literature can be explained with the help of the results obtained from the proposed analytical model.

6.1. Analytical results

The response characteristics given by the analytical results for the lumped mass case with constant wall temperature are plotted in Fig. 2(a). Values typical of an exposed thermocouple bead in a gas turbine were chosen.

Bead properties: diameter = 2 mm, density = 16,600 kg/m^3, emissivity = 1, heat capacity = 162 J/kg K, bead temperature amplitude (T_{bq} = 0.0001 K).

Flow properties: velocity = 10 m/s, pressure = 20 atm, heat transfer coefficient = 1000 W/m^2 K.

Three average body temperatures were chosen to illustrate the increasing effects of radiation. At a low temperature T_{b0} = 300 K there is hardly any effect of radiation. In fact, the asymptotically low frequency response is 0.99391 and has been attenuated by less than one percent. At T_{b0} = 1000 K the asymptotically low frequency response is 0.81513, amounting to a 19% loss. At a still higher temperature T_{b0} = 1600 K, asymptotically low frequency response is 0.51841 with a loss of 48%. Therefore, it is apparent that above a certain break frequency radiative heat transfer no longer plays any significant role in attenuating the response.

6.2. Comparison of numerical and analytical results

The assumption that the wall temperature and the body temperature fluctuations are small relative to their respective averages is best examined by comparing the analytical results with the numerical results obtained for the same input conditions. The numerical and analytical results are nearly in perfect agreement

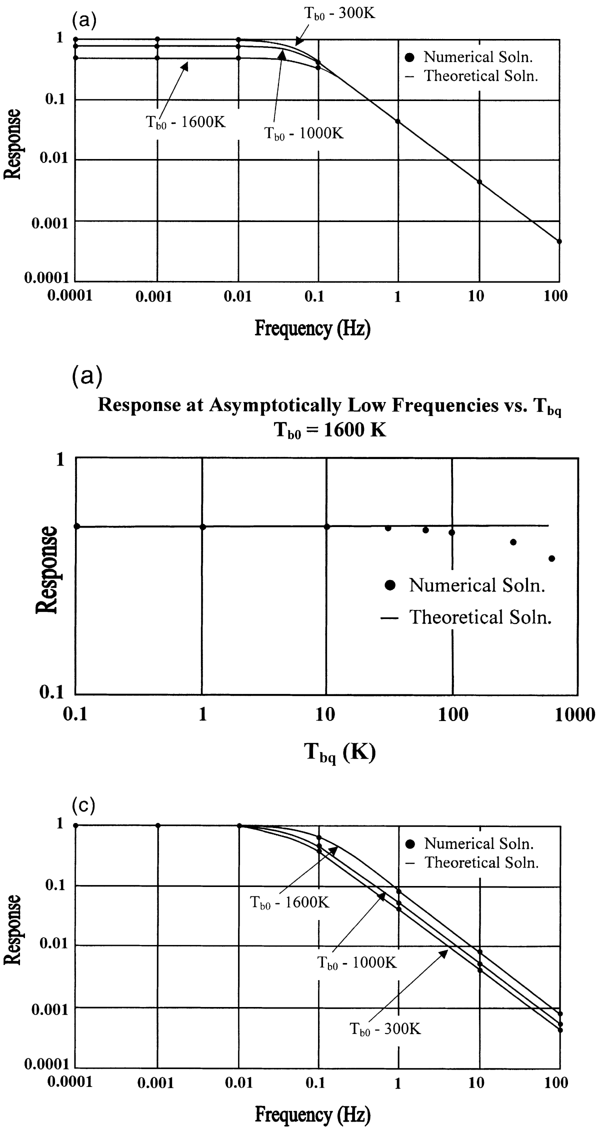

Fig. 2. (a) Temperature response of a lumped mass to a varying gas temperature vs frequency with constant wall temperature. (b) Temperature response of a lumped mass to a varying gas temperature at asymptotically low frequencies vs gas temperature amplitude. (c) Normalized temperature response of a lumped mass to a varying gas temperature vs frequency.

(see Fig. 2(a)) when the body's surface temperature varies by 0.0001 K. At low body temperatures (i.e. $T_{b0} = 300$ K) the effect of radiative heat transfer on the response is so small that any assumption regarding radiative heat transfer becomes of no consequence. Also, above the break frequency where the time response is dominated by thermal lag, the effects of radiation are so small that the assumptions regarding it are trivial. The effect of assuming that the body's temperature variation is small with respect to its average temperature is shown in Fig. 2(b). The input values chosen are kept the same as mentioned earlier with the exception that while T_{b0} is kept at 1600 K, T_{bq} is allowed to vary. A 4% difference is noted between the theoretical and numerical results for asymptotically low frequency response at $T_{bq} = 100$ K, and a 13% difference when $T_{bq} = 300$ K. If one were to accept a 10% difference between the theoretical and numerical result as valid, it can be said that the analytical result is valid when the amplitude of the temperature variation is less than 10% of the average surface temperature of the body (i.e. when $T_{bq}/T_{b0} < 0.1$).

6.3. Normalization of the frequency response

The effects of normalizing the frequency response, as given by Eq. (18), are shown in Fig. 2(c). The effect of normalizing the response to the response at asymptotically low frequencies is to have no effect of radiation at asymptotically low frequencies. Yet, beyond the break frequency, normalizing the response has the effect of decreasing the amount of attenuation with higher temperature. This is the same result as if it was decided to increase the convective heat transfer coefficient by using a linearized radiative–convective heat transfer coefficient. Thus, numerical results which are normalized and compared to theoretical results where a radiative–convective heat transfer result is used, will agree closely.

6.4. Explanation of experimental results and disagreements in the literature

As stated earlier, there is a considerable difference in the literature regarding the effects of radiative heat transfer. Some authors maintain that there is no effect, yet others show a T^3 effect on the time constant. Elmore et al. [8], using a finite difference technique, turned the effect of radiation 'on and off' by setting the emissivity to either 1 or 0, respectively. By comparing their results at three different frequencies, they concluded that there is no effect of radiation on the thermal time constant. However, Sbaibi et al. [9] showed experimental results concluding that the radiative heat transfer does affect the time constant of fine wire thermocouples.

The theoretical results presented by Sbaibi et al. [9] and Malcorps [10] disagree with each other. However, their theoretical results agreed very closely with their own experimental results. The results presented in this

Table 1
Comparison of the low frequency response (responsivity) for some practical conditions

Case	Bead properties			Flow properties			Responsivity
	Mean temperature (K)	Diameter (mm)	Heat capacity (J/kg K)	Velocity (m/s)	Pressure (atm)	Heat transfer coefficient (W/m^2 K)	
Industrial intercooled air compressor exit	300	2	162	10	20	1000	0.994
Gas turbine power system compressor exit	1000	2	162	10	20	1000	0.815
Gas turbine power system turbine inlet	1600	2	162	10	20	1000	0.518
Weather station	288	2	162	10	1	200	0.980
Small scale burner	1650	2	162	40	1	400	0.460

paper agree perfectly with the theoretical results of Malcorps. Also, when normalized, it would agree perfectly with the theoretical results of Sbaibi et al. [9]. Thus the results presented in this paper compare very well with both kinds of reporting in the literature. The disagreement between the findings of different authors in the literature may be explained as follows. Some authors normalized their high frequency results by their low frequency results and concluded that radiation does have an effect but there is no effect of radiation on the thermal time constant of the bodies. They erroneously concluded this because the effect of radiative heat transfer at low frequencies becomes included into their high frequency results by the normalization, while the effect of radiation on the low frequency results is reduced by the normalization. Elmore et al. [8] who performed a numerical study, while correctly concluding that there was no effect of radiation on the time constant, did not show results for asymptotically low frequencies and hence may have missed the effect of radiative heat transfer. Malcorps [11] did correctly show the effect of radiation and did show experimental results that were in agreement with his theoretical results.

6.5. *Effect of time dependent radiation from the walls*

Examination of Eqs. (25) and (26) shows that depending on the value of the phase difference between the object and the walls, the heat transfer can either enhance the response at asymptotically low frequencies if they are in phase, or decrease the probe response time to changes in convective heat transfer if they are out of phase. Therefore, radiation from the walls can interfere with the measurement of the gas temperature fluctuations. In fact, if the net radiative heat transfer from the object to the walls is always equivalent to the net radiative heat transfer from the walls to the object, that is $T_{\mathrm{b0}}^3 T_{\mathrm{b}q} = T_{\mathrm{w0}}^3 T_{\mathrm{w}q}$, and the fluctuations are in phase ($\alpha_{\mathrm{w}q} = \alpha_{\mathrm{b}q}$), radiation has no effect on the response.

6.6. *Relative effect of convection versus radiation in practical systems*

By examination of Eq. (16) it can be seen that the relative effects of convection and radiation are contained in the term

$$\left.\frac{T_{\mathrm{b}q}}{T_{\mathrm{g}q}}\right|_{q\omega_0 \to 0} = \frac{1}{1 + \dfrac{4\varepsilon\sigma T_{\mathrm{b0}}^3}{h}}$$

Malcorps was the first to notice this effect of radiation on the low frequency response. He termed the low frequency response as the 'responsivity' of the sensor. It is interesting to note that his research was in the area of measuring heat flux on the ground from solar radiation and even at those conditions, that is at atmospheric temperatures, responsivity was affected by radiation. If the temperature of the object is high or the convective heat transfer coefficient is low, then the effect of radiation is significant. On the other hand if the convective heat transfer coefficient is very high and the temperature very low then radiative heat transfer has little effect on the frequency response.

As already discussed, in the case of a gas turbine combustor it was found that, in the exit plane of the combustor, the effect of radiative heat transfer is to attenuate the low frequency response by a factor of 0.52. For a weather monitoring station, with typical values [14] of velocity = 10 m/s, temperature = 288 K, pressure = 1 atm and heat transfer coefficient = 200 W/m^2 L, the low frequency temperature response of a thermocouple junction is attenuated by a factor of 0.98; nearly no effect as might be expected. For a small scale industrial burner [15] with velocity = 40 m/s, temperature = 1650 K, pressure = 1 atm and heat transfer coefficient = 400 W/m^2 K, a low frequency response of 0.46 is obtained.

Table 1 presents a summary of the practical cases discussed as above.

Thus it can be seen that at higher temperatures, say greater than 800 K, one should always account for the effect of radiation. In practice this is ordinarily true because the effect of radiation goes as T^3 and quickly makes itself apparent except for the cases of exceptionally high convective heat transfer.

7. Conclusions

Based on the results presented and discussed above, the following conclusions can be drawn.

1. Radiative heat transfer from objects does not affect frequency response past the break frequency; it attenuates the frequency response only for asymptotically low frequencies.
2. If the walls surrounding the object are expected to oscillate with a similar amplitude, phase and average temperature, the net radiative exchange between the walls and the object results in no effect of radiation on the frequency response.
3. For practical problems, to account for the effects of radiation it may be advisable to average over a suitably short time period (i.e. on the order of the break frequency), and ignore the effects of radiation altogether if the walls are exposed to the same fluctuating gas temperatures as the object. However, if there are no walls or the wall temperature is known to be constant (for example, film cooled walls around a gas turbine burner nozzle) relations similar to Eqs. (16) and (17) may be used.
4. The differences in the literature can best be explained by two assumptions often used. The first assumption arises from the desire to linearize the boundary condition by using an approximate combined convective and radiative heat transfer coefficient. This results in a frequency response of unity at asymptotically low frequencies and therefore authors may erroneously conclude that there is an increase in the response time as radiative heat transfer becomes more dominant. The second assumption in the literature, that can explain the confusion, is the desire to normalize the experimentally or numerically determined frequency response to be unity at asymptotically low frequencies. This would explain why numerically derived charts, which should be exact, show the erroneous effect of radiation increasing the response at higher frequencies. Non-normalization of the result shows why Elmore et al. [8] concluded that radiation had no effect on the frequency response above the frequency where thermal lag dominates. It would have been of interest if Elmore et al. had shown any result for asymptotically low frequencies where effect of radiation would be apparent.

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